Representing Relations Using Digraphs

•Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs $(a, b) \in R$ as its edges.

•Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.

•This **one-to-one correspondence** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

Equivalence Relations

- **Equivalence relations** are used to relate objects that are similar in some way.
- **Definition:** A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.
- Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations

•Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a.

•Since R is **reflexive**, every element is equivalent to itself.

•Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

•Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

- **Example:** Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if l(a) = l(b), where l(x) is the length of the string x. Is R an equivalence relation?
- **Solution:**
- R is reflexive, because l(a) = l(a) and therefore aRa for any string a.
- R is symmetric, because if l(a) = l(b) then l(b) = l(a), so if aRb then bRa.
- R is transitive, because if l(a) = l(b) and l(b) = l(c), then l(a) = l(c), so aRb and bRc implies aRc.
- R is an equivalence relation.

Equivalence Classes

- **Definition:** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.
- The equivalence class of a with respect to R is denoted by $[a]_R$.
- When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class.
- If $b \in [a]_R$, b is called a **representative** of this equivalence class.

Equivalence Classes

- **Example:** In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?

- **Solution:** [mouse] is the set of all English words containing five letters.

- For example, ‘horse’ would be a representative of this equivalence class.

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Equivalence Classes

- **Theorem:** Let R be an equivalence relation on a set A. The following statements are equivalent:

- aRb
- [a] = [b]
- $[a] \cap [b] \neq \varnothing$

- **Definition:** A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i,
$i \in I$, forms a partition of S if and only if

(i) $A_i \neq \varnothing$ for $i \in I$

- $A_i \cap A_j = \varnothing$, if $i \neq j$
- $\cup_{i \in I} A_i = S$

Equivalence Classes

• **Examples:** Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition S ?

{{m, o, c, k}, {r, u, b, s}}	yes.
{{c, o, m, b}, {u, s}, {r}}	no (k is missing).
{{b, r, o, c, k}, {m, u, s, t}}	no (t is not in S).
{{u, m, b, r, o, c, k, s}}	yes.
{{b, o, o, k}, {r, u, m}, {c, s}}	yes ({b,o,o,k} = {b,o,k}).
{{u, m, b}, {r, o, c, k, s}, ⍰}	no (⍰ not allowed).

Equivalence Classes

- **Theorem:** Let R be an equivalence relation on a set S. Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Equivalence Classes

- **Example:** Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney.
- Let R be the **equivalence relation** {(a, b) | a and b live in the same city} on the set P = {Frank, Suzanne, George, Stephanie, Max, Jennifer}.
- Then R = {(Frank, Frank), (Frank, Suzanne), (Frank, George), (Suzanne, Frank), (Suzanne, Suzanne), (Suzanne, George), (George, Frank), (George, Suzanne), (George, George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max, Max), (Jennifer, Jennifer)}.

Equivalence Classes

- Then the **equivalence classes** of R are:
- {{Frank, Suzanne, George}, {Stephanie, Max}, {Jennifer}}.
- This is a **partition** of P.

- The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to **exactly one** of the equivalence classes.

Equivalence Classes

- **Another example:** Let R be the relation $\{(a, b) \mid a \equiv b \pmod 3\}$ on the set of integers.
- Is R an equivalence relation?
- Yes, R is reflexive, symmetric, and transitive.
- What are the equivalence classes of R ?
- $\{\{\ldots, -6, -3, 0, 3, 6, \ldots\},$
  $\{\ldots, -5, -2, 1, 4, 7, \ldots\},$
  $\{\ldots, -4, -1, 2, 5, 8, \ldots\}\}$